A BOUND ON GROUP VELOCITY FOR BLOCH WAVE PACKETS

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1. Main result

This short note is a sequel to our previous papers [2], [3] on the asymptotic behavior of Bloch wave packet solutions of the wave equation in periodic media. The purpose is to prove that the group velocity for these Bloch wave packets is bounded by the maximal speed of propagation for the original wave equation. This follows from the fact that the wave packets provide accurate approximate solutions. What was lacking is a mathematical proof that uses only the definition of the group velocity. We follow the notations in [3].

For periodic coefficients $A_0(y)$ and $\rho_0(y)$ in $L^\infty(\mathbb{T}^N)$, consider the Bloch spectral cell problem

$$- (\mathrm{div}_y + 2i\pi\theta)\Big(A_0(y)(\mathrm{grad}_y + 2i\pi\theta)\psi_n\Big) \;=\; \lambda_n(\theta)\rho_0(y)\,\psi_n \quad \text{in} \quad \mathbb{T}^N\,, \tag{1.1}$$

with the Bloch parameter $\theta \in [0,1[^N$ and $\mathbb{T}^N$ the unit torus. Assuming that $A_0(y)$ is symmetric and uniformly coercive and that $\rho_0(y)$ is uniformly bounded away from zero, it is well known that (1.1) admits a countable infinite family of positive real eigenvalues $\lambda_n(\theta)$ (repeated according to their multiplicity) and associated eigenfunctions $\psi_n(\theta, y)$ which, as functions of y, belong to $H^1(\mathbb{T}^N)$ [6, 5, 9, 10]. The eigenvalues, being labeled by increasing order, are Lipschitz functions of θ (not more regular because of possible crossings). However, simple eigenvalues are analytic functions of θ [8]. Being a simple eigenvalue is a generic property [1]. Normalize the eigenfunctions by

$$\int \rho_0(y)\,|\psi_n(y,\theta)|^2\,dy \;=\; 1\,. \tag{1.2}$$

Assumption. Fix $\theta_0 \in [0,1[^N$, $n \in \mathbb{N}$ and assume that $\lambda_n(\theta_0)$ is a **simple** eigenvalue.

Define the associated nonnegative frequency $\omega_n(\theta_0)$ satisfying the dispersion relation,

$$4\,\pi^2\omega_n^2(\theta) \;=\; \lambda_n(\theta)\,. \tag{1.3}$$

The group velocity is then defined by

$$\mathcal{V} \; := \; -\nabla_\theta \, \omega_n(\theta_0) \; = \; \frac{-\nabla_\theta \lambda_n(\theta_0)}{4\pi\sqrt{\lambda_n(\theta_0)}} \, . \tag{1.4}$$

For any fixed $y \in \mathbb{T}^N$ the local speed of propagation is given by

$$c(y) \; = \; \max_{1 \le j \le N} \sqrt{\lambda_j(y)}$$

where $\lambda_j(y)$ are the roots of the characteristic polynomial

$$p(y, \lambda) \; := \; \det\left(A_0(y) - \lambda \rho_0(y) I\right) .$$

The maximal speed of propagation is

$$c_{max} \; := \; \max_{y \in \mathbb{T}^N} c(y) \, .$$

Theorem 1.1. *The group velocity defined by* (1.4) *satisfies*

$$|\mathcal{V}| \; \le \; c_{max} \, .$$

Proof. Introduce the operator

$$\mathbb{A}(\theta)\psi \; := \; -(\mathrm{div}_y + 2i\pi\theta)\Big(A_0(y)(\mathrm{grad}_y + 2i\pi\theta)\psi\Big) - \lambda_n(\theta)\rho_0(y)\psi \, .$$

At the point θ_0, differentiate (1.1) with respect to θ in the direction of the covector ξ to find

$$\mathbb{A}(\theta)\xi.\nabla_\theta\psi_n = 2i\pi\xi \cdot A_0(y)(\nabla_y + 2i\pi\theta)\psi_n + 2i\pi(\mathrm{div}_y + 2i\pi\theta)(A_0(y)\xi\psi_n) + \xi.\nabla_\theta\lambda_n\rho_0(y)\psi_n .$$

Taking the hermitian product of this inequality with ψ_n yields

$$\xi.\nabla_\theta\lambda_n(\theta) \; = \; 2i\pi \int_{\mathbb{T}^N} \Big(\psi_n A_0(y)\xi \cdot \overline{(\nabla_y + 2i\pi\theta)\psi_n} - \overline{\psi_n}\xi \cdot A_0(y)(\nabla_y + 2i\pi\theta)\psi_n\Big) \, dy \, .$$

This implies the upper bound

$$|\xi.\nabla_\theta\lambda_n(\theta)| \; \le \; 4\pi \|\rho_0^{-1/2}A_0^{1/2}\|_{L^\infty(\mathbb{T}^N)} \|\rho_0^{1/2}\psi_n\|_{L^2(\mathbb{T}^N)} \|A_0^{1/2}(\nabla_y + 2i\pi\theta)\psi_n\|_{L^2(\mathbb{T}^N)} \, ,$$

which becomes

$$|\xi.\nabla_\theta\lambda_n(\theta)| \; \le \; 4\pi \, c_{max} \sqrt{\lambda_n(\theta)}$$

as desired. $\square$

Remark 1.2. The proof as given also proves a bound of the propagation speed as a function of direction by the corresponding fastest speeds of the original system (see [4] for an analogous result).

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